# The Voting Paradox

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*A logical explanation as to why perfect democracy is impossible. Without any of that fancy mathematical notation :)*

**Democracy traces its roots to 507 BCE (and potentially even earlier), when, under the leadership of Cleisthenes, the Acropolis of Athens developed a popular rule system.** Adult male citizens over the age of 20 were able to vote for officials that sat in a larger Assembly, a body which made important decisions for the polis. As time passed, democracy remained a viable option for a system of government; however, it was rarely implemented. It was not until the period of Enlightenment, around the 18th century, that democracy became "hyped up" globally. Following the American Revolution (1775–1783) and the French Revolution (1789–1799), systems of democracy began to be implemented on a large scale. It was seen as imperative to ensure *fairness* in society, and especially within the politics which governed that society – but naturally, this raised the question:
*How does a society ensure fair and equal voting?*

## I. What does fair voting look like?

This paper concerns purely democratic voting, with 3 or more candidates. The core ideas behind *fair* voting are: nondictatorship, pareto efficiency, independence of irrelevant alternatives, unrestricted domain, and social ordering.

What does this mean?

- **Nondicatorship:** No single vote should be weighed more than any other vote.
- **Pareto Efficiency (PE):** In the case of unanimity – where every voter favors candidate A over candidate B – candidate A should win.
- **Independence of Irrelevant Alternatives (IIA):** If one choice is removed, the ordering of other choices should not be changed. This is illustrated nicely with a short joke from philosopher Sidney Morgenbesser: "Morgenbesser, ordering dessert, is told by a waitress that he can choose between blueberry or apple pie. He orders apple. Soon the waitress comes back and explains cherry pie is also an option. Morgenbesser replies, 'In that case, I'll have blueberry.'"
- **Unrestricted Domain:** Any voting system should account for every vote casted.
- **Social Ordering:** Candidates should be able to rank choices in any order (as well as potentially indicate ties).

These 5 principles define *Rational Choice Theory (RCT)* – a system which, by following these rules, is intended to be as *fair* as possible. At least, in this context, these conditions fit our societal intuition for what *fair* should look like.

However, in 1950, Kenneth Arrow published his famous "Impossibility Theorem," which proved that it is logically impossible to uphold all 5 of these *ethical* conditions.

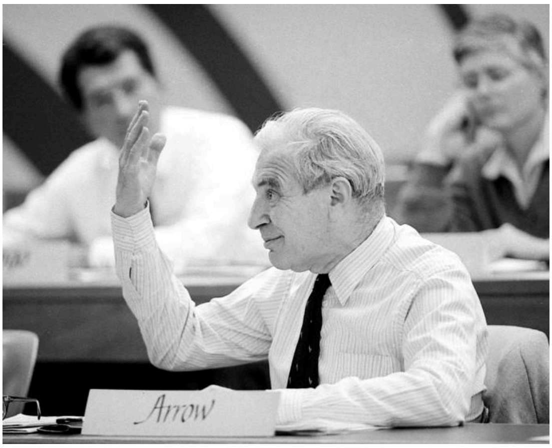


*Image 1: Kenneth J. Arrow (1921–2017)*

## II. Common systems of voting – and Arrow's proof by contradiction

The *simplest* form of democratic voting is what political scientists call "First Past the Post," where every voter casts their ballot for one candidate – and the person with the most votes wins. This system (in combination with the Electoral College and a set of rules defining what happens if no candidate passes 270 electoral votes), is used in the United States for example. However, this clearly violates the IIA condition of RCT. For example, in the 1992 American election: independent Ross Perot (who won zero electoral votes) captured 19% of the popular vote, stealing voters from incumbent George H. W. Bush, ultimately allowing Bill Clinton to win. Put simply, had Perot not run, it is highly likely that Bush would have won another term. Another system is Ranked Voting, where each voter ranks each candidate from 1 to *n*. However, this system can violate the PE condition. For example, take a scenario with 3 voters (A, B, and C) and 3 candidates (X, Y, and Z):

| | 1st choice | 2nd choice | 3rd choice |
|---|---|---|---|
| Voter A | X | Y | Z |
| Voter B | Y | Z | X |
| Voter C | Z | X | Y |

On aggregate, voters prefer X to Y, Y to Z, and Z to X. According to PE, then, all candidates should win? But this is impossible. This cycle of X > Y > Z > X > Y > Z > X > Y > Z > ... is known as the Condorcet Paradox – and therefore proves ranked voting as unviable. Attempts to fix this issue led to the creation of Instant Runoff Voting, a system in which voters rank candidates in order of preference, eliminating candidates with the fewest votes and redistributing those votes until one candidate receives a majority. However, this system violates IIA. For example, take a scenario where:

35% of voters prefer: A > B > C
33% of voters prefer: B > C > A
32% of voters prefer: C > A > B

With all 3 candidates in the race, candidate C will be eliminated after the first round. The votes are then redistributed as follows:

67% of voters prefer: A > B
33% of voters prefer: B > A

And thus, candidate A wins. However, in this situation if candidate B were to withdraw, we would be left with:

35% of voters prefer: A > C
65% of voters prefer: C > A

And thus, candidate C would win. It is now easily shown how the presence or absence of candidate B (the irrelevant alternative) changes the outcome between A and C. Another option, such as Head to Head Voting (in which voters choose a winner between every head to head permutation of candidates) can be shown, by employing Concordet's paradox, to violate RCT. More generally, Arrow's Impossibility Theorem states that any and every proposed *fair and democratic* voting system can be shown to fail to meet all 5 of the conditions of RCT.

## III. Conclusion

Beyond simply being a catchphrase used in the current political climate, "democracy isn't perfect" is a mathematically true statement – at least, when it concerns more than 2 candidates. Every voting system has its own pros and cons, but a trait that each and every one shares is the fact that they all violate RCT in some way. In recent years, however, there have been a number of papers criticizing the conditions required for a voting system to be *fair*. Take for example the Morgenbesser example from earlier. If the options at a restaurant were hamburger and lionfish, I might choose hamburger, knowing how hard it is to properly prepare lionfish. But if a 3rd option, such as pufferfish, is also presented – this changes the situation. Let's say I am allergic to pufferfish – this, by definition, makes it an irrelevant alternative. However, if I know that this restaurant serves more than one kind of exotic fish, I might choose the lionfish, trusting that the chef knows how to prepare and handle exotic fish. Pure formulations of IIA as an axiom only apply to cases where absolutely no additional information is generated by additional options (this is what defines them as "irrelevant"); however, it is often not the case that additional options provide no additional information.

Another common argument against IIA is that cyclical relationships predicate relevance. In other words, some argue that the presence of preferences that form a loop rather than a clear ranking should influence the importance of alternatives. This logic applies to any context where a cyclical relationship might exist. Take for example, the game of Rock Paper Scissors. Let Scissors beat Paper. But if we introduce Rock, now any candidate might come out on top – including Paper. Misapplying IIA here is analogous to declaring that Rock's existence is "irrelevant" to Scissors and Paper. But of course, this could be further from the truth. Rock is incredibly relevant to Scissors and Paper. It is difficult to fathom an interloper more relevant to the Scissors-Paper situation than Rock. All group preferences are potentially cyclical; but importantly, with a large number of voters and candidates, it is unlikely to be a significant factor to any outcome. Considering the game Rock Paper Scissors:

> Scissors beats Paper if there are no other candidates.
>
> Paper beats Rock if there are no other candidates.
>
> And Rock beats Scissors if there are no other candidates.

No matter what the result is of a three-way race, it "violates IIA" by flipping the result of one of these 3. More generally, all voting methods (have a configuration that) violate(s) IIA, because the reality of group preferences violates IIA. Therefore, many argue that IIA should not count as an axiom of RCT, because in reality it will both always exist and simultaneously remain negligible to an outcome.

Overall, through silly examples and simple logic, it is fun and easy to show how *perfect democracy* is impossible.